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# Runaway Modes in Model Field Theories*

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Within the framework of linear quantum field theories, a general study is presented of the existence and removal of runaway modes—solutions of the equations of motion which exhibit a real exponential time dependence. It is hoped that this work will yield insight into the corresponding problem in physically realistic theories. It is shown that runaway modes occur only when the Hamiltonian is not positive definite, and that they occur in linear quantum theories whenever they appear in the corresponding classical theories. Three methods are proposed for eliminating these unphysical modes One is the analog of the method used by Dirac in classical electron theory; the other two are believed to be new.

## I. INTRODUCTION

CERTAIN classical field theories, in particular classical electron theory, have long been known to possess runaway modes[1,2]—solutions of the equations of motion in which observable quantities (typically the electron position) display exponentially increasing time dependence. Recently,[3–5] similar solutions have been found in a number of simple, exactly soluble quantum field theories. In this paper we analyze the properties of runaways in a very simple class of model field theories, linear quantum field theories, for which the Hamiltonian is a quadratic function of the dynamical variables, and the equations of motion linear differential equations. All of the quantum models in which runaways have been discovered have been of this class. We are not interested in these models for their own sake, but because we hope that information gained from them may cast light upon the properties of runaways in more complicated, and more realistic, theories. Likewise, we attempt to devise general methods to alter these theories in such a way as to eliminate the runaways but leave the more desirable aspects of the theories intact, because we hope that these methods may be generalized to more complicated theories. Unlike theories afflicted with some other pathological conditions, such as ghost states,[6,7] theories with runaways are internally consistent and obey the general principles of quantum mechanics: probability and energy are conserved, the inner product is positive definite, etc. We seek to remove runaways only because the behavior they describe is not in agreement with the observed behavior of physical systems.

In Sec. II we discuss the general properties of runaway solutions in linear theories and prove two theorems concerning their existence. In Sec. III, in order to clarify these general considerations, we show how they apply to two specific model theories. Section IV contains three methods for the removal of runaways; each of these is realized in one or both of the specific models considered in Sec. II. One of these methods is the quantum analog of the method proposed by Dirac[8] for classical electron theory. We believe the other two methods are new. Because of the simplicity of our models, the three methods yield almost the same results. Work in progress, however, indicates that they lead to quite different modified theories when applied to more complicated models. To display the difference between the methods more clearly, we briefly discuss in Sec. V the three types of response to an external force which arise when a particular theory with runaways is modified in each of our three ways. Section VI is a discussion of the results.

## II. GENERAL PROPERTIES OF RUNWAYS

The theories we will discuss have the following dynamical variables: A set of $N$ Hermitian field operators $\phi_i(\mathbf{x},t)$, their canonical momenta $\pi_i(\mathbf{x},t)$; a set of $M$ single-particle position operators $r_i(t)$, their canonical momenta $p_i(t)$. These operators obey the equal-time commutation rules,

$$[\phi_i(\mathbf{x},t),\pi_j(\mathbf{x}',t)]=i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}'), \tag{1}$$

$$[r_i(t),p_j(t)]=i\delta_{ij}. \tag{2}$$

We are concerned only with linear field theories whose Hamiltonians are quadratic in the field variables.

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These Hamiltonians together with the associated linear field equations can also be interpreted as describing a set of classical theories where the field variables are $c$ numbers. We call these the corresponding $c$-number theories.

The general solution of a corresponding $c$-number theory can be represented as a (infinite) sum of linearly independent solutions, each with an arbitrary coefficient. The same is true in the quantum theory except that the arbitrary coefficients are replaced by operators determined (to within a unitary transformation) by the commutation rules. If the Hamiltonian does not depend explicitly upon the time, we can choose each member of this sum to have an exponential time dependence. The solutions then have the form

$$\begin{aligned} \phi_j(\mathbf{x},t)&=\textstyle\sum_\alpha f_{\alpha j}(\mathbf{x})O_\alpha e^{-i\omega_\alpha t}+\text{H.c.},\\ \pi_j(\mathbf{x},t)&=\textstyle\sum_\alpha g_{\alpha j}(\mathbf{x})O_\alpha e^{-i\omega_\alpha t}+\text{H.c.},\\ r_j(t)&=\textstyle\sum_\alpha c_{\alpha j}O_\alpha e^{-i\omega_\alpha t}+\text{H.c.},\\ p_j(t)&=\textstyle\sum_\alpha d_{\alpha j}O_\alpha e^{-i\omega_\alpha t}+\text{H.c.}, \end{aligned} \tag{3}$$

where the explicit sum over $\alpha$ includes all frequencies $\omega_\alpha$ such that $\text{Re}\,\omega_\alpha\geq 0$. The terms H.c. then contain all solutions for which $\text{Re}\,\omega_\alpha\leq 0$.

In the most familiar cases the frequencies $\omega_\alpha$ are all real and the modes oscillatory. However, there exist theories in which some of the $\omega_\alpha$ have imaginary parts; these are the so-called runaway modes. If we denote by a superscript $s$ ($s$ for sensible) those parts of the field operators which contain no runaway modes, we may write

$$\phi_j(\mathbf{x},t)=\phi_j{}^s(\mathbf{x},t)+[\textstyle\sum_\alpha f_{\alpha j}(\mathbf{x})O_\alpha e^{-i\omega_\alpha t}+\text{H.c.}], \tag{4}$$

where the sum includes only those frequencies for which $\text{Im}\,\omega_\alpha\neq 0$. We may similarly decompose the other dynamical variables.

The structure of the Hamiltonian also follows from general considerations. It is a time-independent, quadratic function of the dynamical variables and hence must have the form

$$H=\textstyle\sum_\alpha h_\alpha O_{-\alpha}O_\alpha+\tilde{h}_\alpha O_\alpha O_{-\alpha}, \tag{5}$$

where $O_{-\alpha}$ belongs with $-\omega_\alpha$. From the time independence of Eqs. (2) and (3), we can conclude that $O_\alpha$ commutes with every $O_\beta$ except $O_\beta=O_{-\alpha}$, and from

$$i[H,O_\alpha]=-i\omega_\alpha O_\alpha, \tag{6}$$

we have

$$\omega_\alpha=(h_\alpha+\tilde{h}_\alpha)[O_\alpha,O_{-\alpha}]. \tag{7}$$

If we now adjust the phases and the normalizations of the $O_\alpha$ to satisfy

$$[O_\alpha,O_{-\alpha}]=\omega_\alpha/|\omega_\alpha|, \tag{8}$$

then, to within the addition of a real, $c$-number constant,

$$H=\tfrac{1}{2}\textstyle\sum_\alpha|\omega_\alpha|(O_{-\alpha}O_\alpha+O_\alpha O_{-\alpha}). \tag{9}$$

Again, as in Eq. (4), we can separate the contribution from the non-runaway modes and write

$$H=H^s+\tfrac{1}{2}\textstyle\sum_\alpha{}'|\omega_\alpha|(O_{-\alpha}O_\alpha+O_\alpha O_{-\alpha}), \tag{10}$$

where the primed sum runs only over the frequencies with a nonvanishing imaginary part. In the remainder of this paper the explicit appearance of the $O_\alpha$ will refer only to the runaway modes.

We now prove two simple theorems on the occurrence of runaway modes.

(*a*) *Runaways occur in a linear field theory if and only if the energy spectrum is a continuum extending from minus infinity to plus infinity.* If the Hamiltonian is positive definite, there can be no runaways.

*Proof.* We use the decomposition of the Hamiltonian of Eq. (10). If there are no runaways, this is a sum of positive definite terms. If there are runaways, the spectrum of the runaway part can be shown[4] to extend from minus infinity to plus infinity. Since the runaway part is dynamically independent of the rest of the Hamiltonian, the spectrum of the total Hamiltonian must have the same property.

(*b*) *Whenever the c-number equations of motion admit a runaway solution, it must be present in the quantum theory, provided only that the c-number solution is of finite energy.* [It is clear that something like this proviso is needed, for otherwise we would be forced to include, in free meson theory, solutions of the Klein-Gordon equation of the form $\exp(i\mathbf{k}\cdot\mathbf{x}-i\omega t)$ with $k$ and $\omega$ imaginary.]

*Proof.* Assume the theorem is false. Then there exists a solution of the quantum theory that is free from runaways. But we can obtain another solution of the quantum theory by adding to this one a $c$-number runaway solution of the equations of motion; since the equations of motion are linear, this is a solution, and the addition of a $c$ number does not change the commutation rules. This new solution obeys the same equations of motion and canonical commutation rules as the old one, so they must be connected by a time-independent unitary transformation. (Actually, it is well known[9] that this is not strictly the case in field theory; it is also necessary that the transformation considered only cause a finite change in the energy. The conditions we have imposed are sufficient to insure this for quadratic Hamiltonians.) But this is inconsistent with the representation of the solution given by Eq. (3).

## III. TWO THEORIES WITH RUNWAYS

### A. Pair Theory

The pair theory is a model of meson-nucleon interactions which was invented by Wentzel[10] to study nuclear forces. It describes the interaction of a scalar meson field $\phi(\mathbf{x},t)$ with a fixed source characterized by a spherically symmetric form factor $\rho(\mathbf{x})$, normalized such

[9] L. van Hove, Physica **18**, 145 (1952).
[10] G. Wentzel, Helv. Phys. Acta **15**, 111 (1942).

that $\int \rho(\mathbf{x})d^3x=1$. It is defined by the Hamiltonian

$$H=\frac{1}{2}\int d^3x[(\nabla\phi)^2+\mu^2\phi^2+\pi^2] +\tfrac{1}{2}g_0\left[\int \rho(\mathbf{x})\phi(\mathbf{x},t)d^3x\right]^2, \quad (11)$$

and the commutation rules

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]=i\delta^3(\mathbf{x}-\mathbf{x}'). \quad (12)$$

These yield the canonical equations of motion,

$$\dot{\phi}=i[H,\phi]=\pi, \quad (13)$$

$$\dot{\pi}=i[H,\pi]=(\nabla^2-\mu^2)\phi-g_0\rho\int\rho\phi d^3x,$$

which in turn imply the field equation,

$$(\Box^2-\mu^2)\phi(\mathbf{x},t)=g_0\rho(\mathbf{x})\int\rho(\mathbf{x}')\phi(\mathbf{x}',t)d^3x'. \quad (14)$$

If $g_0$ is positive, the solution of Eq. (14) is

$$\phi(\mathbf{x},t)=\phi_0(\mathbf{x},t)+\frac{1}{(2\pi)^2}\int d^4k\, e^{i\mathbf{k}\cdot\mathbf{x}}\frac{\rho(\mathbf{k})}{(k_0^2-\omega_k^2)D(k_0)} \times\int d^3k'\,\rho(\mathbf{k}')\phi_0(\mathbf{k}',k_0), \quad (15)$$

where $\omega_k$ is $(\mathbf{k}^2+\mu^2)^{\frac{1}{2}}$, $D(k_0)$ is a Green's function, and $\phi_0$ a solution of the homogeneous Klein-Gordon equation. In particular, if $D$ is either the retarded function $D_R$ or the advanced function $D_A$, defined by

$$D_{R,A}(k_0)=\frac{1}{g_0}-\int\frac{\rho^2(\mathbf{k})d^3k}{(k_0\pm i\epsilon)^2-\omega_k^2}, \quad (16)$$

where

$$\rho(\mathbf{k})=\frac{1}{(2\pi)^{\frac{3}{2}}}\int\rho(\mathbf{x})e^{i\mathbf{k}\cdot\mathbf{x}}d^3x, \quad (17)$$

then $\phi_0$ is either $\phi_{\text{in}}$, the asymptotic field operator in the far past, or $\phi_{\text{out}}$, the asymptotic field operator in the far future.[11]

Since we can express $\phi$ in terms of both $\phi_{\text{in}}$ and $\phi_{\text{out}}$, we may eliminate $\phi$ from these two equations and obtain an expression for the scattering matrix.[12] We shall not write this out explicitly, but only state that the low-momentum scattering of mesons off the source is characterized not by $g_0$ but by $g_r$, the renormalized coupling constant.

$$\frac{1}{g_r}=\frac{1}{g_0}+4\pi\int_0^\infty\rho^2(k)dk. \quad (18)$$

If $g_r$ is kept fixed as the nucleon becomes a point source $[\rho(\mathbf{x})\rightarrow\delta^3(\mathbf{x});\rho(\mathbf{k})\rightarrow(2\pi)^{-\frac{3}{2}}]$, $g_0$ goes to negative infinity. We may write $D$ in terms of $g_r$.

$$D_{R,A}=\frac{1}{g_r}-4\pi(k_0^2-\mu^2)\int\frac{\rho^2(k)dk}{(k_0\pm i\epsilon)^2-\omega_k^2}. \quad (19)$$

At some stage during this limiting process, $g_0^{-1}$ becomes negative. After this point, the Hamiltonian (11) is no longer positive definite; it has a continuous spectrum extending over the entire real axis. According to theorem (a) of Sec. II, the theory must then contain runaways, and, indeed, just at this point Eq. (14) admits two additional solutions of the form

$$\phi(\mathbf{x},t)=\phi^s(\mathbf{x},t)+\frac{p(\mathbf{x})}{(2\omega_0)^{\frac{1}{2}}}(O_1e^{-\omega_0t}-O_{-1}e^{\omega_0t}), \quad (20)$$

which must be added to the sensible solution $\phi^s$ of Eq. (15) if we are to preserve the canonical commutators. That such a runaway solution must be included in the expression for the meson field follows from theorem (b) of Sec. II. The function $p(\mathbf{x})$ in Eq. (20) is equal to

$$p(\mathbf{x})=N_p\int d^3x'\frac{\exp[-|\mathbf{x}-\mathbf{x}'|(\omega_0^2+\mu^2)^{\frac{1}{2}}]}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}'), \quad (21)$$

where $N_p$ is adjusted so that $\int p^2d^3x=1$. The imaginary runaway frequency $i\omega_0$ is the root of the equation

$$D_{RA}(i\omega_0)=0. \quad (22)$$

It can be readily be verified from Eq. (16) that such a root only occurs if $g_0$ is negative.

### B. Dirac Harmonic Oscillator[13]

This second theory describes the electric dipole interaction of a radiation field with a charged, spinless particle bound in a simple harmonic potential of spring constant $K$. It is defined by the Hamiltonian

$$H=\frac{\left[\mathbf{p}-e\int\rho(\mathbf{x})\mathbf{A}(\mathbf{x},t)d^3x\right]^2}{2m_0} +\tfrac{1}{2}K\mathbf{r}^2+\frac{1}{2}\int d^3x[(\nabla\mathbf{A})^2+\boldsymbol{\pi}^2], \quad (23)$$

and the commutation rules

$$[r_i(t),p_j(t)]=i\delta_{ij}, \quad (24)$$

$$[A_i(\mathbf{x},t),\pi_j(\mathbf{x}',t)]=i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}'). \quad (25)$$

Equation (25) is not the proper commutator for the vector potential; it is inconsistent with the gauge

[11] C. N. Yang and D. Feldman, Phys. Rev. **105**, 1378 (1957).

[12] R. E. Norton and A. Klein, Phys. Rev. **109**, 584 (1958); see also A. Klein and B. McCormick, Phys. Rev. **98**, 1428 (1955).

[13] For more details, see reference 4. Our notation is close to that of this reference.

condition.[2] However, the theory with the correct gauge condition can also be solved and differs from this model in none of the features of interest here. To avoid complication, we consider this simplified model.

The function $\rho(\mathbf{x})$ is a form factor which will eventually approach a delta function, just as in the pair theory. We only outline the development of this theory here; it parallels closely that of the pair theory and has been discussed extensively in the literature.[2–4]

The canonical equations of motion are

$$d\mathbf{r}/dt=\frac{1}{m_0}\left(\mathbf{p}-e\int\rho\mathbf{A}d^3x\right),$$

$$d\mathbf{A}/dt=\boldsymbol{\pi},$$

$$d\mathbf{p}/dt=-K\mathbf{r},$$

$$d\boldsymbol{\pi}/dt=\nabla^2\mathbf{A}+\frac{e\rho}{m_0}\left(p-e\int\rho\mathbf{A}d^3x\right), \tag{26}$$

which lead to the field equations

$$m_0\ddot{\mathbf{r}}+K\mathbf{r}=-e\int\rho\mathbf{A}d^3x, \tag{27}$$

$$\Box^2\mathbf{A}=-e\rho\dot{\mathbf{r}}. \tag{28}$$

When $m_0$ is positive these equations have only the sensible solutions $r^s$ and $A^s$. The scattering amplitude can be constructed in the same way as in the pair theory, and one is led to define a mass renormalization

$$m_r=m_0+4\pi e^2\int_0^\infty\rho^2(k)dk. \tag{29}$$

As in the pair theory, the renormalization integral becomes infinite in the limit of a local interaction, and the unrenormalized quantity—here the bare mass, there the reciprocal of the bare coupling constant—tends to negative infinity. When $m_0$ is negative the Hamiltonian in Eq. (23) is no longer positive definite; its spectrum extends over the entire real axis. The equations of motion again admit runaway solutions,

$$\mathbf{r}(t)=\mathbf{r}^s(t)-\frac{C}{(2u_0)^{\frac{1}{2}}}(\mathbf{O}_1e^{-u_0t}+\mathbf{O}_{-1}e^{u_0t}), \tag{30}$$

$$\mathbf{A}(\mathbf{x},t)=\mathbf{A}^s(\mathbf{x},t)+\frac{h(\mathbf{x})}{(2u_0)^{\frac{1}{2}}}(\mathbf{O}_1e^{-u_0t}-\mathbf{O}_{-1}e^{u_0t}), \tag{31}$$

which must be added to the sensible solutions $r^s$ and $A^s$. The function $h(\mathbf{x})$ has the form

$$h(\mathbf{x})=N_h\int d^3x'\,\frac{e^{-u_0|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x}-\mathbf{x}'|}\rho(\mathbf{x}'), \tag{32}$$

where $N_h$ is such that $\int h^2d^3x=1-KC^2u_0^{-2}$. The imaginary frequency $iu_0$ is the root of the equation

$$G(iu_0)=-m_0u_0^2-K-e^2u_0^2\int\frac{d^3k\,\rho^2(\mathbf{k})}{u_0^2+\mathbf{k}^2}=0, \tag{33}$$

and the parameter $C$ is related to the residue of $G^{-1}$ at this singularity

$$-C^2\equiv\lim_{\omega\to iu_0}\frac{\omega^2+u_0^2}{G(\omega)}. \tag{34}$$

In analogy with the situation in the pair theory, Eq. (33) has a root only if $m_0$ is negative.

In both models, the runaway solutions, characterized by $O_1$ and $O_{-1}$,

$$[O_1,O_{-1}]=-i, \tag{35}$$

only occur when a parameter in the Hamiltonian, either the bare coupling constant or the bare mass, assumes such a value that the spectrum of the Hamiltonian becomes a bottomless continuum. For fixed values of the renormalized mass and coupling constant, this unavoidably happens as the cutoff is removed.

### IV. REMOVAL OF RUNAWAYS

In this section we consider some methods of modifying our model theories in such a way as to produce new theories in which runaways do not occur. There are many such modifications that we could make; for example, we could simply eliminate the interaction altogether, but this sort of procedure is clearly unsatisfactory—what we want is a method of removing the runaways that in some sense does minimal damage to the original theory. We want the modified theory to look like the original theory except for small times and high energies.

The idea of minimal modification is not sharply defined in general, but can be given a precise meaning in linear theories. In a linear theory, we can demand that our modification does not alter the time dependence or the commutation rules of the sensible part of the dynamical variables, that it affect the runaway part only. (We can do this because the sensible and runaway parts are dynamically independent; this is not true in a nonlinear theory.)

A quantum theory is normally specified by the Hamiltonian and the canonical commutators. These imply the Heisenberg equations of motion. Alternatively, we may specify the theory by giving these equations of motion and the commutation rules at a fixed time. This determines the Hamiltonian to within a $c$-number constant. Our first method of modification is based on the latter formulation.

#### A. Modification of the First Kind

This is the direct analog of the method used by Dirac[8] to remove the runaways from classical electrodynamics. We maintain without change the original Heisenberg equations of motion, but modify the commutation rules by removing from them the contribution from the runaway modes. From Eqs. (3), (4), and (8), we see that a typical member of the new set of truncated commutators is

$$[\phi_i(x,t),\pi_j(\mathbf{x}',t)] = i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{x}')-\sum_\alpha{}' [f_{\alpha i}(\mathbf{x})g_{-\alpha j}(\mathbf{x}') - f_{-\alpha i}(\mathbf{x})g_{\alpha j}(\mathbf{x}')]\omega_\alpha/|\omega_\alpha|, \quad (36)$$

where the primed sum extends only over the runaway modes. Similar expressions apply for the other commutators. In particular, we note that $[\phi_i(x,t),\phi_j(x',t)]$, etc., no longer in general vanish.

For the pair theory,[3] Eq. (36) becomes

$$[\phi(\mathbf{x},t),\pi(\mathbf{x}',t)]=i\delta^3(\mathbf{x}-\mathbf{x}')-ip(\mathbf{x})p(\mathbf{x}'), \quad (37)$$

and, for the oscillator theory,[2–4]

$$[A_i(\mathbf{x},t),\pi_j(\mathbf{x}',t)]=i\delta_{ij}[\delta^3(\mathbf{x}-\mathbf{x}')-h(\mathbf{x})h(\mathbf{x}')]. \quad (38)$$

The truncated commutators, together with the old equations of motion, have for a solution the sensible part of the dynamical variables only, so the theory modified in this manner is indeed free from runaways.

Many of the properties of the canonical commutators are not shared by the truncated commutators. For example, it is not true that they determine the dynamical variables to within a unitary transformation; for, if they did, we could apply unaltered the arguments of theorem (b) of Sec. II to show that the solution must contain a runaway. This means that the modified theory is not yet completely defined—to any solution we may add an arbitrary $c$-number runaway and obtain a new solution, not unitarily equivalent.

To remove this ambiguity, we must add to the theory an asymptotic condition that suppresses the runaways. The modified theory of the first kind is then completely defined by (1) the original Heisenberg equations of motion, (2) the truncated commutators, and (3) an asymptotic condition—that the matrix elements of the dynamical variables do not increase exponentially in the far past or future.

In many cases, as a consequence of the equations of motion, the sensible parts of the dynamical variables are asymptotic in the far past to a set of free dynamical variables, the "in" operators. The "in" operators obey the free equations of motion and the canonical commutators; since they are functions of the sensible parts only, removal of the runaways does not alter this. Both our examples have this property. In this case, we may substitute for (2) and (3) above:

(2′) An asymptotic condition on the past—that the dynamical variables are asymptotic in the past to a set of "in" operators that obey the canonical commutators.

(3′) An asymptotic condition on the future—that the matrix elements of the dynamical variables do not increase exponentially in the far future.

### B. Modification of the Second Kind

In this scheme we retain the original Hamiltonian (that is, the form of the Hamiltonian as a function of the dynamical variables) instead of the original equations of motion. The modified theory of the second kind is defined by (1) the original Hamiltonian function and (2) the truncated commutators. The field equations then assume a different form than in the original theory, but they still admit the sensible parts of the dynamical variables as solutions. The Hamiltonian is now positive definite to within a $c$ number, so this method does indeed remove the runaways. Since this theory uses the truncated commutators, it shares the ambiguity of the modified theory of the first kind, and a subsidiary condition is required to remove it. However, in this case, the lack of uniqueness is only to the addition of a time-independent $c$-number function.

We will work out the actual equations in the case of the pair theory. Applying the truncated commutators (37) to the Hamiltonian (11), we find, after considerable operator algebra,

$$\dot\phi=\pi-p\int p\pi d^3x,$$
$$\dot\pi=(\nabla^2-\mu^2)\phi-g_0\rho\int\rho\phi d^3x-\omega_0^2p\int p\phi d^3x. \quad (39)$$

From these relations, together with Eqs. (21) and (22), we deduce

$$\int p\dot\phi d^3x=\int p\dot\pi d^3x=0. \quad (40)$$

Hence,

$$C_1=\int p\phi d^3x,$$
$$C_2=\int p\pi d^3x \quad (41)$$

are constant $c$ numbers. With these definitions, the general solution to the field Eqs. (39) is

$$\phi=\phi^s+C_1p(\mathbf{x}),$$
$$\pi=\pi^s+C_2p(\mathbf{x}), \quad (42)$$

where $\phi^s$ is given in Eq. (15). As mentioned, the former runaways are here constant $c$-number functions with arbitrary coefficients.

### C. Modification of the Third Kind

In this scheme we add an additional term to the Hamiltonian while retaining the canonical commutators. If this additional term involves only the runaway operators, it cannot interfere with the propagation of the sensible parts of the dynamical variables. If it is chosen such that the part of the Hamiltonian involving the $O$'s becomes positive definite, it must suppress the runaways. It is clear there are many ways to add such terms; in order to preserve the linearity of our models we add quadratic terms only. In particular, we consider a change in the Hamiltonian of the form $H\rightarrow H+\delta H$, where

$$\delta H=\lambda \textstyle\sum_\alpha' |\omega_\alpha| (O_\alpha O_\alpha^\dagger + O_{-\alpha}O_{-\alpha}^\dagger). \tag{43}$$

If $\lambda\geq\frac{1}{2}$, the part of the Hamiltonian associated with each $|\omega_\alpha|$ is a positive definite quadratic form; the new theory is without runaways.

In the oscillator model,

$$\delta H=\lambda C^2\left[\left(\mathbf{p}-\frac{u_0}{c}\int h\mathbf{A}d^3x\right)^2 +\left(\frac{K\mathbf{r}}{u_0}+\frac{1}{C}\int h\pi d^3x\right)^2\right]. \tag{44}$$

Note that the additional terms only involve the fields close to the particle, with a characteristic distance of the classical electron radius. At first glance, it might appear surprising that we are able to remove such a grossly macroscopic phenomenon as the runaway by an alteration of the Hamiltonian at small distances. Actually, the macroscopic character of the runaway is illusory. At any fixed time, the runaway has the spatial dependence of a Yukawa function, is always concentrated strongly about the electron position. (This is also true in such non-linear theories as classical electron theory.) Thus, from a Hamiltonian viewpoint, the runaway is always a microscopic phenomenon.

Since this theory uses the canonical commutators, it does not share the ambiguities of theories with truncated commutators. However, there is a new difficulty. The former runaways have become true oscillators, "tremblings of the field in the neighborhood of the particle," "internal electron degrees of freedom." In order to specify a motion, we need not only specify the "in" operators, but also the state of the $O_\alpha$, the internal state of the particle. Because the internal state does not influence the scattering, (the $O_\alpha$ do not appear in the $S$ matrix), this is not a serious problem. However, we can always consider the limit of the theory as $\lambda$ goes to infinity. In this case the tremblings of the field becoming increasingly rapid, and the first excited state of the internal oscillator becomes increasingly removed from its ground state. In the limit, the electron must be in its internal ground state for any process taking place at finite energy.

## V. THE EXTERNAL FORCE

A great deal of insight into the structure of a physical system may be gained by studying its response to an external force. In this section, we describe the response of the oscillator model to a delta-function impulse applied at $t=0$, in the original theory and in the three modified theories. We can split the solution into a $q$-number solution of the homogeneous equations of motion and a $c$-number solution of the inhomogeneous equations with $c$-number force. This split is not unique; we choose to do it in such a way that the $c$-number solution vanishes in the remote past. (Retarded response.)

In the original theory and in the modified theories of the second and third kind, the force may be introduced by adding an additional term $\mathbf{e}_1\cdot\mathbf{r}\delta(t)$ to the Hamiltonian. The modified theory of the first kind is not formulated in terms of a Hamiltonian; therefore we introduce a force by adding a term $e_1\delta(t)$ to the third of Eqs. (26).

We characterize the response by giving the $c$-number part of the $x$ component of the electron velocity as a function of time. For simplicity, we set the spring constant equal to zero and the cutoff equal to a delta function. The characteristic frequency of the system, $u_0$, is then simply $m/e^2$.

The calculations are uninstructive and of much the same kind as those done earlier. We omit them and merely give the results.

### A. Original Theory

$$\begin{aligned} v&=0, && t\leq 0\\ &=(1-e^{u_0t})/m, && t\geq 0. \end{aligned} \tag{45}$$

The external force induces a runaway.

### B. Modified Theory of the First Kind

$$\begin{aligned} v&=e^{u_0t}/m, && t\leq 0\\ &=1/m, && t\geq 0. \end{aligned} \tag{46}$$

This is the acausal response first noticed by Dirac[8] in classical electron theory. This quantum theory has an unusual property: there is no Hamiltonian. That is to say, there is no unitary operator connecting the dynamical variables at different times; they belong to inequivalent representations of the truncated commutators. However, although there is no $U$ matrix, there is still an $S$ matrix. Thus we may say that although there is no conservation of probability over short times, in the long run, probability is conserved.

### C. Modified Theory of the Second Kind

The propagation of the system for times less than zero must be determined by the Hamiltonian for that time, which is the free Hamiltonian. Thus the response here cannot show the acausality displayed by the theory of the first kind.

$$\begin{aligned} v&=0, && t\leq 0\\ &=(1-e^{-u_0t})/m, && t\geq 0. \end{aligned} \tag{47}$$

### D. Modified Theory of the Third Kind

$$\begin{aligned} v&=0, \quad t\leq 0\\ &=\frac{(1-e^{-u_0t})}{m}-\frac{\sin[u_0t(4\lambda^2-1)^{\frac{1}{2}}]}{m(4\lambda^2-1)^{\frac{1}{2}}}, \quad t\geq 0. \end{aligned} \tag{48}$$

As $\lambda$ goes to infinity, this becomes the response of the theory of the second kind, in this sense the limit of the theory of the third kind. It is easy to see that this is also true for more general perturbations than a $c$-number force. The entities which enter into the perturbation formula for the $S$ matrix, for example, are quantities like $T(A(\mathbf{x},t),A(\mathbf{x}',t'))$. If the perturbation is slowly varying over times of order $(u_0\lambda)^{-1}$, we may replace such expressions by their averages over such times. When we do this, the contribution from the rapidly varying former runaway parts vanishes, and all that remains is the contribution from the sensible parts of the dynamical variables. That is to say, for high $\lambda$, the Green's functions may be replaced by the Green's functions for the modified theory of the second kind; therefore the response must approach the response of the modified theory of the second kind.

## VI. DISCUSSION

We have found three different methods that succeed in removing the runaways from linear field theories while retaining their desirable features. The three kinds of modified theories produced by these methods are all acausal; however, they are acausal in slightly different senses. One definition of the word causal, and the one used most often in classical physics, is that the system responds to an external force only after the force is turned on. Only the modified theory of the first kind is acausal in this sense. Another definition is that the commutators for separate spatial points vanish at equal times (microcausality). The modified theories of the first and second kind are acausal in this sense. Another definition is that the analytic continuation of the scattering amplitude has no poles in the upper half plane. All three kinds of modified theory are acausal in this sense.

It is interesting to speculate on the possibility of extending these methods to more complicated models. The first method is certainly the most elegant and unambiguous. However, that it yields a consistent theory is something of a miracle; we have no reason for believing that this will always occur—it just happens to work for linear theories. If it works for more complicated models it must be because of some general principle which we do not now know, and which might be of considerable importance.

Our third method of modification, on the other hand, can certainly always be extended. We can always do *something* to the Hamiltonian that will remove the runaways. Here the interesting question is a quantitative one: how little damage can we do to the other features of the theory while removing the runaways? For linear theories we can escape with essentially no damage. Again, we would be surprised if this were the case for more complicated models.

With regard to the second method, the primary question is a more primitive one. We do not have any idea on how to extend this method. Certainly trying all possible noncanonical commutators does not seem to be a fruitful approach. Perhaps the remarks in the last paragraph of Sec. V offer some clues, but the method they suggest seems difficult and indirect.

In conclusion, we emphasize that the problem that motivates this inquiry is still unsolved. We do not know if runaways (or, indeed, any of the pathological conditions which are observed in model theories) occur in realistic field theories such as quantum electrodynamics. Their occurrence in simple models may be an indication of their general occurrence. If runaways are present in realistic field theories, they are a symptom of the inadequacy of the physical ideas on which these theories are founded. A study of how to remove the runaways without altering the low-energy behavior of the theory may lead to an understanding of the missing physics.

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